The central limit theorem for multiplicative systems

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1. ALEXITS introduced the following definitions (see [1], p. 186): The sequence of real measurable functions $\varphi_1(t), \varphi_2(t), \dots$ defined in the interval $[0, 1]$ is called a multiplicative system if all its finite products are Lebesgue integrable with

$$(1) \qquad \int_0^1 \varphi_{n_1}(t) \dots \varphi_{n_k}(t)\, dt = 0 \qquad (n_1 < \dots < n_k;\ k = 1, 2, \dots).$$

The sequence $\{\varphi_n(t)\}$ is called an equinormed strongly multiplicative system (in abbreviation: ESMS) if the system $\{\varphi_{n_1}(t) \dots \varphi_{n_k}(t)\}$ $(n_1 < \dots < n_k; k = 1, 2, \dots)$ is orthonormal, i.e.

$$(2) \qquad \int_0^1 \varphi_n(t)\, dt = 0, \quad \int_0^1 \varphi_n^2(t)\, dt = 1 \qquad (n = 1, 2, \dots);$$

$$\int_0^1 \varphi_{n_1}^{r_1}(t) \dots \varphi_{n_k}^{r_k}(t)\, dt = \int_0^1 \varphi_{n_1}^{r_1}(t)\, dt \dots \int_0^1 \varphi_{n_k}^{r_k}(t)\, dt,$$

where $r_1, \dots, r_k$ can be equal to 1 or 2.

2. These notions proved to be tractable and useful ones, because the behaviour of the series arising from the functions of an ESMS resembles, in many respects, that of series of independent functions. This is not suprising, as a sequence of independent functions with mean value 0 and dispersion 1 is an ESMS. Another example for ESMS, also having a lot of properties in common with the independent functions, is a strongly lacunary sequence of trigonometric functions, i.e. $\{\sqrt{2} \sin 2\pi m_k x\}$, where $m_{k+1}/m_k \geqq 3$ $(k = 1, 2, \dots)$.

A number of authors have generalized the central limit theorem for lacunary trigonometric series. The most general result is due to SALEM and ZYGMUND [11], who state the following

Theorem A. *Let $S_N(t)$ denote the* N*th partial sum of the lacunary trigonometric series* $\sum_{k=1}^{\infty} (a_k \cos m_k t + b_k \sin m_k t)$, $m_{k+1}/m_k \geqq q > 1$ $(k = 1, 2, \dots)$, *and let*

$a_1, a_2, \ldots; b_1, b_2, \ldots$ be arbitrary sequences of real numbers for which

$$C_N = \left\{\frac{1}{2}\sum_{k=1}^{N}(a_k^2+b_k^2)\right\}^{1/2} \to \infty \quad \textit{and} \quad \{a_N^2+b_N^2\}^{1/2} = o(C_N).$$

Then for any set $E \subset [0, 2\pi]$ of positive measure the distribution functions

$$F_N(y; E) = \frac{\operatorname{mes}(\{t: \in E: S_N(t)/C_N < y\})}{\operatorname{mes}(E)} \;{}^{1})$$

tend pointwise to the Gaussian distribution with mean value 0 *and dispersion* 1.

The present author (see [6]) managed to generalize the above theorem almost word by word to ESMS. Before saying it in an explicit form, we introduc the notations

$$S_N(t) = \sum_{n=1}^{N} c_n \varphi_n(t), \quad C_N^2 = \sum_{n=1}^{N} c_n^2,$$

where $\{c_n\}$ is an arbitrary sequence of real numbers.

Theorem B. *Let $\{\varphi_n(t)\}$ be a uniformly bounded* ESMS. *If*

(3) $\quad$ (i) $C_N \to \infty$ $\quad$ *and* $\quad$ (ii) $c_N = o(C_N)$,

then the distribution functions

$$F_N(y) = \operatorname{mes}\left(\left\{t \in [0, 1]: \frac{S_N(t)}{C_N} < y\right\}\right)$$

tend pointwise to the Gaussian distribution function

$$G(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{t^2}{2}}\, dt.$$

Another generalization of the central limit theorem is due to Révész [10], whose theorem reads as follows:

Theorem C. *Let $\{\varphi_n(t)\}$ be a uniformly bounded multiplicative system such that*

(4) $$\int_0^1 \varphi_m^2(t)\varphi_n^2(t)\, dt = \int_0^1 \varphi_n^2(t)\, dt = 1 \quad (m \neq n).$$

If

$$C_N \to \infty \quad \textit{and} \quad \sum_{N=2}^{\infty} \frac{c_N^4}{C_N^4} \log^2 N < \infty,$$

then for every y, $F_N(y)$ tends to the Gaussian distribution function.

3. In this paper we propose to give a complete solution of the problem of the central limit theorem concerning uniformly bounded ESMS or only multiplicative

[1]) mes (E) denotes the Lebesgue measure of a set E.

systems satisfying (4). We shall prove an even more general result, namely, that in the case of ESMS the distribution function of $S_N(t)/C_N$ on every fixed set of positive measure tends to the Gaussian distribution function.

Theorem 1. *Let* $\{\varphi_n(t)\}$ *be a uniformly bounded* ESMS, *and let E be a subset of* $[0, 1]$ *with* $\operatorname{mes}(E)>0$. *If* (3) *holds then*

$$F_N(y; E) = \frac{\operatorname{mes}\{(t\in E: S_N(t)/C_N < y)\}}{\operatorname{mes}(E)}$$

tends pointwise to the Gaussian distribution function.

We should like to point out that in case $E=[0, 1]$ the central limit theorem is valid under apparently weaker conditions than those of Theorem 1.

Theorem 2. *Let* $\{\varphi_n(t)\}$ *be a uniformly bounded multiplicative system satisfying* (4). *If* (3) *holds then* $F_N(y)$ *tends pointwise to the Gaussian distribution function.*

Evidently this theorem contains, as particular cases, both Theorem B and Theorem C.

The theorem which follows indicates that if $C_N\to\infty$ then the second condition (3) is indispensable for the validity of Theorem 1. We note that by a distribution function we mean any non-decreasing function $F(y)$, continuous to the left, with $\lim\limits_{y\to-\infty} F(y)=0$ and $\lim\limits_{y\to+\infty} F(y)=1$.

Theorem 3. *Let* $\{\varphi_n(t)\}$ *be a uniformly bounded* ESMS, *and let* $E\subset[0, 1]$ *be a set of positive measure. Suppose that* $C_N\to\infty$ *and that* $F_N(y; E)$ *tends to a distribution function* $F(y)$ *(at the points of continuity of the latter) such that either* $F(y)>0$ *or* $F(y)<1$ *for all finite* y. *Then* (3) (ii) *must hold.*

Theorem 1 and Theorem 3 generalize the results of Salem and Zygmund [11] from lacunary trigonometric series to ESMS.

4. Suppose now that

$$\sum_{n=1}^{\infty} c_n\varphi_n(t) \tag{5}$$

is of the class L^2, i.e. $\sum\limits_{n=1}^{\infty} c_n^2<\infty$. It is known that in this case the series (5) converges almost everywhere in $[0, 1]$. (See Alexits [2], or Alexits and Tandori [3]). The remainder $\sum\limits_{n=N}^{\infty} c_n\varphi_n(t)$ of (5) represents then a certain function $R_N(t)$. By $G_N(y; E)$ we shall mean the distribution function of $R_N(t)/D_N$ over the set $E\subset[0, 1]$ of positive measure, where $D_N^2=\sum\limits_{n=N}^{\infty} c_n^2$. The proofs of the following two results are repetitions of those of Theorem 1 and Theorem 3.

Theorem 4. *Let* $\{\varphi_n(t)\}$ *be a uniformly bounded* ESMS, *and let* E *be a set with* $\operatorname{mes}(E)>0$. *Suppose that* (5) *is of the class* L^2, *and that* $c_N/D_N \to 0$. *Then* $G_N(y; E)$ *tends to the Gaussian distribution function.*

Theorem 5. *Let* $\{\varphi_n(t)\}$ *be a uniformly bounded* ESMS, *and let* E *be a set of positive measure. If* (5) *is of the class* L^2 *and if* $G_N(y; E)$ *tends to a distribution function* $F(y)$ *which is not constant outside a finite interval then* $c_N/D_N \to 0$.

Instead of the partial sums of (5) we may consider linear means of the partial sums. Analogous theorems are valid for ESMS just as much as for lacunary trigonometric series, and we refer again to the cited paper of SALEM and ZYGMUND [11]. To make the analogy complete, let us mention the following fact, which plays an important role in the considerations of the above authors: If the series (5) is summable by a linear summation process, regular in the sense of Toeplitz, in a set of positive measure then (5) must converge almost everywhere. (See in [8] of the present author.)

5. Finally we remark that the notion of multiplicativity can be much more generally defined as follows (as for this terminology, see, in detail, RÉVÉSZ [9]): Let $\{\Omega, \mathscr{S}, \mathbf{P}\}$ be a probability space, and let $\xi_1, \xi_2, \dots$ be a sequence of random variables on Ω with $E(\xi_n)=0$, $E(\xi_n^2)=1$ $(n=1, 2, \dots)$, where $E(\xi)$ means the expectation of ξ.[2] The sequence $\xi_1, \xi_2, \dots$ is then called, for example, an ESMS if

$$E(\xi_{n_1}^{r_1}\dots\xi_{n_k}^{r_k}) = E(\xi_{n_1}^{r_1})\dots E(\xi_{n_k}^{r_k}) \qquad (n_1<\dots<n_k;\ k=2, 3, \dots),$$

where $r_1, \dots, r_k$ can be equal to 1 or 2.

In this paper we consider only the particular case, when $\Omega=[0, 1]$, $\mathscr{S}$ is the class of Lebesgue measurable subsets of [0, 1], and $\mathbf{P}$ is the common Lebesgue measure on it. Our results hold in the general case as well, without changing the proofs, but terms must be replaced by suitable ones in the probability theory.

6. To prove Theorem 1 and Theorem 2 our point of departure is the following lemma which is of interest in itself.

Lemma 1. *Let* $\{\varphi_n(t)\}$ *be a uniformly bounded system of real functions satisfying* (4). *If* (3) *holds then*

$$\frac{1}{C_N^2}\sum_{n=1}^{N} c_n^2\varphi_n^2(t)$$

converges in measure to 1.

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[2]) I.e. $E(\xi) = \int\limits_{\Omega} \xi(\omega)\, d\mathbf{P}$.

We note that this is an earlier result of the present author (see [7]), but there was given an unnecessarily complicated proof. That is why we give another proof, much simpler than the former one.

Proof of Lemma 1. We begin with

$$\frac{1}{C_N^2}\sum_{n=1}^{N} c_n^2\varphi_n^2(t) = 1+\frac{1}{C_N^2}\sum_{n=1}^{N} c_n^2(\varphi_n^2(t)-1) = 1+\xi_N(t).$$

We observe that owing to (4) the system $\{\varphi_n^2(t)-1\}$ is orthogonal[3]), on the other hand, it is not difficult to see that (3) is equivalent to

$$(\max_{1\leqq n\leqq N}|c_n|)/C_N \to 0. \tag{6}$$

Hence the measure of the set of points where $|\xi_N(t)|\geqq\varepsilon>0$ is less than

$$\frac{1}{\varepsilon^2}\int_0^1 \xi_N^2(t)\,dt = \frac{1}{\varepsilon^2}\frac{1}{C_N^4}\sum_{n=1}^{N} c_n^4\left\{\int_0^1 \varphi_n^4(t)\,dt-1\right\}$$

and so tends to 0 on account of (6) and the uniform boundedness of $\{\varphi_n(t)\}$. Thus the proof of Lemma 1 is complete.

7. Proofs of Theorem 1 and Theorem 2. The main idea used throughout the proof is due to SALEM and ZYGMUND (see [11]).

We make use of the classical method of characteristic functions. In view of Paul LÉVY's theorem it is enough to show that over any finite range of x the characteristic function of $F_N(y; E)$ tends to that of the Gaussian distribution with mean value 0 and dispersion 1, i.e. to $e^{-\frac{1}{2}x^2}$. The characteristic function of $F_N(y; E)$ is

$$\Phi_N(x) = \int_{-\infty}^{+\infty} e^{-ixy}\,dF_N(y; E).$$

From the definition of the Lebesgue integral we find that

$$\Phi_N(x) = \frac{1}{\operatorname{mes}(E)}\int_E \exp\left\{-\frac{ixS_N(t)}{C_N}\right\}dt = \frac{1}{\operatorname{mes}(E)}\int_E \prod_{n=1}^{N}\exp\left\{-\frac{ixc_n\varphi_n(t)}{C_N}\right\}dt. \tag{7}$$

Hence, using the fact that

$$\exp z = (1+z)\exp\{\tfrac{1}{2}z^2+O(|z|^3)\} = (1+z)\exp\{\tfrac{1}{2}z^2+o(|z|^2)\}$$

[3]) P. RÉVÉSZ called our attention to the fact that this can make proofs of theorems concerning ESMS simpler.

for $z \to 0$, we can write (7) in the form

$$\text{(8)} \qquad \frac{1}{\operatorname{mes}(E)} \int_E e^{o(1)} \prod_{n=1}^{N} \left\{ \left(1 - \frac{ixc_n \varphi_n(t)}{C_N}\right) \exp\left(-\frac{x^2 c_n^2 \varphi_n^2(t)}{2C_N^2}\right) \right\} dt,$$

where the term $o(1)$ in $e^{o(1)}$ tends to 0 uniformly in t as $N \to \infty$, provided $x = O(1)$, which we assume from now on.

Observe now (since $1 + u \leqq e^u$) that

$$\left| \prod_{n=1}^{N} \left(1 - \frac{ixc_n \varphi_n(t)}{C_N}\right) \right| \leqq \left\{ \prod_{n=1}^{N} \left(1 + \frac{x^2 c_n^2 K^2}{C_N^2}\right) \right\}^{1/2} \leqq e^{\frac{1}{2}x^2 K^2},$$

where K denotes a common bound of the system $\{\varphi_n(t)\}$. Hence, by virtue of Lemma 1, it follows that, with an error tending uniformly to 0 as $N \to \infty$, the integral (8) is

$$\text{(9)} \qquad \frac{1}{\operatorname{mes}(E)} e^{-\frac{1}{2}x^2} \int_E \prod_{n=1}^{N} \left(1 - \frac{ixc_n \varphi_n(t)}{C_N}\right) dt.$$

Denote the last integral by I_N. It is enough to show that I_N tends to mes (E).

By (1) this is immediate if $E = [0, 1]$ and thus the proof of Theorem 2 is complete.

Continuing the proof of Theorem 1, expand the integrand of (9) in the form

$$\prod_{n=1}^{N} \left(1 - \frac{ixc_n \varphi_n(t)}{C_N}\right) = 1 + \sum_{\nu \geqq 1} \alpha_\nu^{(N)} \psi_\nu(t),$$

where the numbers $\alpha_\nu^{(N)}$ depending also on x and the product system $\{\psi_\nu(t)\}$ are defined as follows: Let

$$\nu = 2^{n_1} + \cdots + 2^{n_k} \qquad (0 \leqq n_1 < \cdots < n_k; \quad k \geqq 1)$$

denote the dyadic representation of the index $\nu \geqq 1$. Set

$$\alpha_\nu^{(N)} = \begin{cases} \left(-\dfrac{ix}{C_N}\right)^k c_{n_1+1} \cdots c_{n_k+1} & \text{if} \quad 1 \leqq \nu < 2^N, \\ 0 & \text{if} \quad \nu \geqq 2^N; \end{cases}$$

and

$$\psi_\nu(t) = \varphi_{n_1+1}(t) \dots \varphi_{n_k+1}(t).^{4)}$$

By (2) it is obvious that the system $\{\psi_\nu(t)\}$ is orthonormal. In particular, we obtain

$$I_N = \operatorname{mes}(E) + \sum_{\nu \geqq 1} \gamma_\nu \alpha_\nu^{(N)},$$

4) $\{\psi_\nu(t)\}$ is called the W-system generated by $\{\varphi_n(t)\}$. (See in detail ALEXITS [1], p. 187.)

where the γ_ν are the Fourier coefficients of the characteristic function of E with respect to the system $\{\psi_\nu(t)\}$. In view of (6) each $\alpha_\nu^{(N)}$ tends to 0 as $N\to\infty$ ($\nu=1, 2, \ldots$). Hence, if ν_0 is fixed,

(10) $$\sum_{\nu\geqq 1} |\gamma_\nu \alpha_\nu^{(N)}| = \sum_{\nu\leqq\nu_0} + \sum_{\nu>\nu_0} \leqq o(1)+\Big(\sum_{\nu>\nu_0} \gamma_\nu^2\Big)^{1/2}\Big(\sum_{\nu>\nu_0} |\alpha_\nu^{(N)}|^2\Big)^{1/2}.$$

The first factor in the last product is arbitrarily small if ν_0 is large enough (since $\Sigma\gamma_\nu^2<\infty$), and we are going to show that the second factor is bounded. To estimate it we remove the restriction that $\nu>\nu_0$. This adds only non-negative terms to it, and the expression obtained in this way then collapses back into

$$\prod_{n=1}^{N}\left(1+\frac{x^2 c_n^2}{C_N^2}\right).$$

Taking into account again that $1+u\leqq e^u$, we can see that this product is not greater than e^{x^2}. Collecting results we deduce that the second factor in the last product of (10) is bounded. Hence $I_N\to\text{mes}\,(E)$, and this concludes the proof of Theorem 1.

8. Proof of Theorem 3. We note that the proof does not require the full power of a uniformly bounded ESMS but only a much weacker assumption on $\{\varphi_n(t)\}$, namely, that it is a uniformly bounded system, say with K.

The following argument follows closely that of a similar theorem in the paper of Salem and Zygmund [11]. For the sake of completeness, we give the proof in detail here too.

Let us assume for example $F(y)<1$ for all finite y, and that (3) (ii) is false. Then there is an $\varepsilon>0$ such that $c_N/C_N>\varepsilon$ for infinitely many k; consider only such k. Let $E_N(y)$ denote the subset of E where $S_N(t)/C_N<y$. Obviously, $C_{N-1}/C_N<(1-\varepsilon^2)^{1/2}$ and the formula

$$\frac{S_N(t)}{C_N}=\frac{S_{N-1}(t)}{C_{N-1}}\cdot\frac{C_{N-1}}{C_N}+\frac{c_N\varphi_N(t)}{C_N}$$

shows that at every point t of $E_{N-1}(y)$ with $y>0$ we have

$$\frac{S_N(t)}{C_N}<y(1-\varepsilon^2)^{1/2}+K.$$

It follows that $E_{N-1}(y)$ is included in $E_N\big(y(1-\varepsilon^2)^{1/2}+K\big)$, and so

$$F_{N-1}(y;E)\leqq F_N\big(y(1-\varepsilon^2)^{1/2}+K;E\big).$$

Let y be a point of continuity of $F(y)$. Letting $N\to\infty$ gives

(11) $$F(y)\leqq F\big(y(1-\varepsilon^2)^{1/2}+K\big).$$

However, from the assumption that $F(y)$ is always less than 1 and from the fact that $F(y)\to 1$ as $y\to\infty$ it follows that there are points of continuity $y>0$ such that

$$F(y) > F\big(y(1-\varepsilon^2)^{1/2}+K\big).$$

This contradicts (11) and completes the proof of Theorem 3.

9. Finally, we produce an application which shows that the central limit theorems can be used to obtain exact estimates on the asymptotic behaviour of $S_N(t)$ in L^p norm ($p>0$). In [6] the present author has already proved that the mean of degree p of $S_N(t)$ and the C_N are equal to within a factor. To be more precise, our cited theorem reads as follows:

Theorem D. *Let $\{\varphi_n(t)\}$ be a uniformly bounded* ESMS. *Then, for every positive real number p, we have*

$$K_1^{(p)} C_N \leqq \left\{\int_0^1 |S_N(t)|^p\,dt\right\}^{1/p} \leqq K_2^{(p)} C_N, \tag{12}$$

where $K_1^{(p)}$ and $K_2^{(p)}$ are positive constants depending only on p.

Now we are able to improve these inequalities not only in case [0, 1] but for an arbitrary fixed set $E\subset[0, 1]$ of positive measure when (3) is fulfilled. Namely, we show the following holds:

Theorem 6. *Let $\{\varphi_n(t)\}$ be a uniformly bounded* ESMS, *let $E\subset[0,1]$ be a set of positive measure, and let p be a positive number. If* (3) *holds then*

$$\frac{1}{C_N^p}\int_E |S_N(t)|^p\,dt \to \frac{\operatorname{mes}(E)}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}|x|^p e^{-\frac{x^2}{2}}\,dx. \tag{13}$$

This theorem makes the relation (12) much more exact in case (3). As to its proof we refer to a known result concerning the convergence of distribution functions and that of their moments.

Lemma 2. *Let p and δ be positive numbers, and let $F(y), F_1(y), F_2(y), \ldots$ be the distribution functions of the $L^{p+\delta}$ integrable functions $f(t), f_1(t), f_2(t), \ldots$ with*

$$\int_0^1 |f_n(t)|^{p+\delta}\,dt \leqq K \qquad (n = 1, 2, \ldots),$$

where K means a positive constant. If $F_n(y)$ converges to $F(y)$ at the points of continuity of the latter then

$$\int_0^1 |f_n(t)|^p\,dt \to \int_0^1 |f(t)|^p\,dt.$$

This lemma can be found, for example, in the book of LOÈVE ([4], pp. 178—185) or, in the special case $p=1$, it is proved in our paper [7]. The proof given in the latter can be extended to this more general case word for word, but Schwarz's inequality must be replaced by Hölder's inequality in it.

To prove Theorem 6, after having Lemma 2, it suffices to remark that

$$\int_E |f(t)|^p \, dt = \operatorname{mes}(E) \int_{-\infty}^{+\infty} |y|^p \, dF(y; E),$$

where $F(y; E)$ denotes the distribution function of $f(t)$ relative to the set E.

As to the integral on the right-hand side of (13), with the factor $1/\sqrt{2\pi}$, can be easily calculated in many cases, for example, it is equal to $1.3 \ldots (p-1)$ if p is an even number.

It would be interesting to know whether Theorem 6 remains valid also in case the integrand on the left-hand side of (13), or only the integrand of (12), is substituted by $\max\limits_{1 \leq n \leq N} |S_n(t)|$. The analogous result for independent functions is true (see [5]). Such a result, if valid, would considerably contribute to the investigation of the divergence features of the series arising from the functions of an ESMS.

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